Problem Set 12

Macroeconomics III

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Fall 2021

Problem

We consider a model for the Danish economy described by:

$$\pi_t = m_t \tag{1}$$

$$x_t = \theta_t + (\pi_t - \pi_t^e) - \epsilon_t \tag{2}$$

$$\mathbb{E}\left[L(\pi_t, x_t)\right] = \frac{1}{2}\Big(\mathbb{E}\left[(\pi_t - \bar{\pi})^2\right] + \lambda^D \mathbb{E}\left[(x_t - \bar{x})^2\right]\Big) \tag{3}$$

Where $\bar{\pi} = 0$. Agents know θ_t when they form expectations, whereas the monetary authority determines m_t after observing ϵ_t and θ_t.

Furthermore, we assume that monetary policy in the eurozone follow,

$$\pi_t^{eu} = \frac{1}{1 + \lambda^{eu}} \epsilon_t^{eu} \tag{4}$$

Finally, we note that

$$\theta_t \sim i.i.d(0, \sigma_\theta^2), \quad \epsilon_t \sim i.i.d(0, \sigma_\epsilon^2), \quad \epsilon_t^{eu} \sim i.i.d(0, \sigma_{eu}^2), \quad \sigma_\epsilon^2 > \sigma_{eu}^2$$

a) - Optimal Policy Rule - Set-Up

As a benchmark find the optimal monetary policy rule and the equilibrium inflation and output outcomes.

$$
\begin{aligned}
\pi_t &= m_t \\
x_t &= \theta_t + (\pi_t - \pi_t^e) - \epsilon_t \\
\mathbb{E}\left[L(\pi_t, x_t)\right] &= \frac{1}{2}\left(\mathbb{E}\left[\pi_t^2\right] + \lambda^D \mathbb{E}\left[(x_t - \bar{x})^2\right]\right) \\
\pi_t^{eu} &= \frac{1}{1+\lambda^{eu}}\epsilon_t^{eu}
\end{aligned}
$$

We note that the loss-function is quadratic, why we know that the optimal policy rule is a linear function in the shocks.

$$\pi_t^C = \psi + \psi_\theta \theta_t + \psi_\epsilon \epsilon_t$$

Find:

1. π_t
2. x_t

a) - Optimal Policy Rule (1/3)

As a benchmark find the optimal monetary policy rule and the equilibrium inflation and output outcomes.

We note that the loss-function is quadratic, why we know that the optimal policy rule is a linear function in the shocks.

$$\pi_t^C = \psi + \psi_\theta \theta_t + \psi_\epsilon \epsilon_t$$

The agents are aware of this and form their expectations accordingly

$$\pi_t^e = \mathbb{E}[\psi + \psi_\theta \theta_t + \psi_\epsilon \epsilon_t | \theta_t] = \psi + \psi_\theta \theta_t$$

The difference between actual and expected inflation is:

$$\pi_t^C - \pi_t^e = \psi_\epsilon \epsilon_t$$

Output is then

$$x_t^C = \theta_t + (\pi_t^C - \pi_t^e) - \epsilon_t = \theta_t - (1 - \psi_\epsilon)\epsilon_t$$

a) - Optimal Policy Rule (2/3)

The monetary authority minimizes the expected loss,

$$
\begin{aligned}
\mathbb{E}[L] &= \frac{1}{2}\left\{\mathbb{E}\left[(\pi_t^C)^2\right] + \lambda^D \mathbb{E}\left[(x_t^C - \bar{x})^2\right]\right\} \\
&= \frac{1}{2}\left\{\mathbb{E}\left[(\psi + \psi_\theta \theta_t + \psi_\epsilon \epsilon_t)^2\right] + \lambda^D \mathbb{E}\left[(\theta_t - (1-\psi_\epsilon)\epsilon_t - \bar{x})^2\right]\right\}
\end{aligned}
$$

We know that $\mathbb{E}[\theta \cdot \epsilon] = \mathbb{E}[\theta \cdot \psi] = \mathbb{E}[\psi \cdot \epsilon] = 0$.

$$
= \frac{1}{2}\left\{\psi^2 + \psi_\theta^2 \mathbb{E}[\theta_t^2] + \psi_\epsilon^2 \mathbb{E}[\epsilon_t^2] + \lambda^D \left(\mathbb{E}[\theta_t^2] + (1-\psi_\epsilon)^2 \mathbb{E}[\epsilon_t^2] + \bar{x}^2\right)\right\}
$$

Additionally, due to zero mean we know that $\mathbb{E}[\theta^2] = \sigma_\theta^2$ and $\mathbb{E}[\epsilon^2] = \sigma_\epsilon^2$

$$
= \frac{1}{2}\left\{\psi^2 + \psi_\theta^2 \sigma_\theta^2 + \psi_\epsilon^2 \sigma_\epsilon^2 + \lambda^D \left(\sigma_\theta^2 + (1-\psi_\epsilon)^2 \sigma_\epsilon^2 + \bar{x}^2\right)\right\}
$$

We now have the expected loss as a function of the monetary policy rule and the variances.

a) - Optimal Policy Rule (3/3)

The expected loss is minimized wrt. ψ, ψ_θ and ψ_ϵ.

$$\frac{1}{2}\left\{\psi^2 + \psi_\theta^2\sigma_\theta^2 + \psi_\epsilon^2\sigma_\epsilon^2 + \lambda^D\left(\sigma_\theta^2 + (1-\psi_\epsilon)^2\sigma_\epsilon^2 + \bar{x}^2\right)\right\}$$

FOC wrt. ψ:

$$\psi = 0$$

FOC wrt. ψ_θ:

$$\psi_\theta\sigma_\theta^2 = 0$$

FOC wrt. ψ_ϵ:

$$\begin{aligned}\psi_\epsilon\sigma_\epsilon^2 &= \lambda^D(1-\psi_\epsilon)\sigma_\epsilon^2\\ \psi_\epsilon &= \lambda^D(1-\psi_\epsilon)\\ \psi_\epsilon &= \frac{\lambda^D}{1+\lambda^D}\end{aligned}$$

We have the optimal monetary policy rule

$$\pi_t^C = \psi + \psi_\theta\theta_t + \psi_\epsilon\epsilon_t = \frac{\lambda^D}{1+\lambda^D}\epsilon_t$$

The corresponding output is

$$x_t^C = \theta_t - (1-\psi_\epsilon)\epsilon_t = \theta_t - \frac{1}{1+\lambda^D}\epsilon_t$$

b) - Anachich Party - Set-Up

Candidates from a anarchic party announce that they will make the Central Bank behave according to the needs of the Ministry of Finance (i.e. they would not be able to work independently). Estimate inflation and output outcomes under this administration. Is monetary policy announced by the Central Bank credible?

Not credible $\implies$ discretionary monetary policy

$$\begin{aligned} \pi_t &= m_t \\ x_t &= \theta_t + (\pi_t - \pi_t^e) - \epsilon_t \\ L(\pi_t, x_t) &= \frac{1}{2}\left(\pi_t^2 + \lambda^D (x_t - \bar{x})^2\right) \\ \pi_t^{eu} &= \frac{1}{1+\lambda^{eu}} \epsilon_t^{eu} \end{aligned}$$

Find:

1. π_t
2. x_t

b) - Discretion (1/2)

Estimate inflation and output outcomes under discretionary monetary policy. Is monetary policy announced by the Central Bank credible?

We insert the expression for x_t into the loss-function

$$L(\pi_t, x_t) = \frac{1}{2}\left[\pi_t^2 + \lambda^D(\theta_t + \pi_t - \pi_t^e - \epsilon_t - \bar{x})^2\right]$$

FOC wrt. π_t is:

$$\pi_t^D = -\lambda^D(\theta_t + \pi_t^D - \pi_t^e - \epsilon_t - \bar{x})$$
$$\pi_t^D = \frac{\lambda^D}{1+\lambda^D}(\pi_t^e + \epsilon_t + \bar{x} - \theta_t)$$

We find expectations

$$\pi_t^e = \frac{\lambda^D}{1+\lambda^D}\left(\pi_t^e + \underbrace{\mathbb{E}[\epsilon_t|\theta_t]}_{=0} + \bar{x} - \theta_t\right) = \frac{\lambda^D}{1+\lambda^D}(\pi_t^e + \bar{x} - \theta_t)$$
$$\pi_t^e = \lambda^D(\bar{x} - \theta_t)$$

b) - Discretion (2/2)

We now find the optimal inflation and output.

$$\begin{aligned}
\pi_t^D &= \frac{\lambda^D}{1+\lambda^D}(\pi_t^e + \epsilon_t + \bar{x} - \theta_t) \\
&= \frac{\lambda^D}{1+\lambda^D}\Big(\underbrace{\lambda^D(\bar{x} - \theta_t)}_{\pi_t^e} + \epsilon_t + \bar{x} - \theta_t\Big) \\
&= \lambda^D(\bar{x} - \theta_t) + \frac{\lambda^D}{1+\lambda^D}\epsilon_t
\end{aligned}$$

Note: $\lambda^D(\bar{x} - \theta_t)$ is the inflation bias from discretion policy - intuitively, it can be seen as the result of a failed attempt at stabilising wrt. θ_t.

We can then find output

$$\begin{aligned}
x_t^D &= \theta_t + (\pi_t - \pi_t^e) - \epsilon_t \\
&= \theta_t + \underbrace{\lambda^D(\bar{x} - \theta_t) + \frac{\lambda^D}{1+\lambda^D}\epsilon_t}_{\pi_t} - \underbrace{\lambda^D(\bar{x} - \theta_t)}_{\pi_t^e} - \epsilon_t \\
&= \theta_t - \frac{1}{1+\lambda^D}\epsilon_t
\end{aligned}$$

c) - Currency-Peg - Set-Up

A very conservative candidate probably would adopt the euro. Estimate the inflation and growth rates under this administration (i.e. when there is simple rule that fixes the inflation rate to the eurozone one)

$$\pi_t = m_t$$

$$x_t = \theta_t + (\pi_t - \pi_t^e) - \epsilon_t$$

$$\mathbb{E}\left[L(\pi_t, x_t)\right] = \frac{1}{2}\left(\mathbb{E}\left[\pi_t^2\right] + \lambda^D \mathbb{E}\left[(x_t - \bar{x})^2\right]\right)$$

$$\pi_t^{eu} = \frac{1}{1 + \lambda^{eu}} \epsilon_t^{eu}$$

Find:

1. π_t
2. x_t

c) - Currency Peg

A very conservative candidate probably would adopt the euro. Estimate the inflation and growth rates under this administration (i.e. when there is simple rule that fixes the inflation rate to the eurozone one)

The monetary policy will be set equal to that of the eurozone. The inflation is, therefore:

$$\pi_t^{eu} = \frac{1}{1+\lambda^{eu}}\epsilon_t^{eu}$$

The expectations will be

$$\pi_t^{e} = \mathbb{E}\left[\frac{1}{1+\lambda^{eu}}\epsilon_t^{eu}\right] = 0$$

The output will then be

$$x_t^{eu} = \theta_t + \pi_t - \pi_t^e - \epsilon_t = \theta_t + \frac{1}{1+\lambda^{eu}}\epsilon_t^{eu} - \epsilon_t$$

d) - Simple-Rule - Set-Up

Estimate the inflation and growth rates under this rule, under the assumption that it is a linear rule (i.e. $m = a + b\theta$). Find the optimal parameters a and b.

$$
\begin{aligned}
\pi_t &= m_t \\
x_t &= \theta_t + (\pi_t - \pi_t^e) - \epsilon_t \\
\mathbb{E}\left[L(\pi_t, x_t)\right] &= \frac{1}{2}\left(\mathbb{E}\left[\pi_t^2\right] + \lambda^D \mathbb{E}\left[(x_t - \bar{x})^2\right]\right) \\
\pi_t^{eu} &= \frac{1}{1+\lambda^{eu}}\epsilon_t^{eu} \\
\pi_t^S &= a + b\theta_t
\end{aligned}
$$

Find:

1. π_t
2. x_t

d) - Simple Rule

Estimate the inflation and growth rates under this rule, under the assumption that it is a linear rule (i.e. $m = a + b\theta$). Find the optimal parameters a and b.

Monetary policy, expectations and output are then

$$\pi_t^S = a + b\theta_t, \qquad \pi_t^e = a + b\theta_t, \qquad x_t^S = \theta_t - \epsilon_t$$

The expected loss is given by

$$\begin{aligned}\mathbb{E}\left[L\right] &= \frac{1}{2}\left\{\mathbb{E}\left[(a + b\theta_t)^2\right] + \lambda^D \mathbb{E}\left[(\theta_t - \epsilon_t - \bar{x})^2\right]\right\} \\ &= \frac{1}{2}\left\{a^2 + b^2\sigma_\theta^2 + \lambda^D\left(\sigma_\theta^2 + \sigma_\epsilon^2 + \bar{x}^2\right)\right\}\end{aligned}$$

FOCs are $a = 0$ and $b = 0$. The rule that minimizes is $(a, b) = (0, 0)$.

$$\pi_t^S = 0, \qquad x_t^S = \theta_t - \epsilon_t$$

Note: Since expectations are correct, $\pi = \pi^e$, inflation is incapable of stabilizing output. Hence, inflation only increases the loss, why they set it to zero.

e) - Central Banker - Set-Up

Your favourite candidate would choose to have an independent Central Bank, choosing the preferences of the central banker (λ^B) in an optimal way. Estimate the inflation and growth rates.

$$\pi_t = m_t$$

$$x_t = \theta_t + (\pi_t - \pi_t^e) - \epsilon_t$$

$$\mathbb{E}\left[L(\pi_t, x_t)\right] = \frac{1}{2}\left(\mathbb{E}\left[\pi_t^2\right] + \lambda^D \mathbb{E}\left[(x_t - \bar{x})^2\right]\right)$$

$$\pi_t^{eu} = \frac{1}{1+\lambda^{eu}}\epsilon_t^{eu}$$

$$\pi_t^B = \lambda^B(\bar{x} - \theta_t) + \frac{\lambda^B}{1+\lambda^B}\epsilon_t$$

Find:

1. π_t
2. x_t
3. Optimal λ^B - is it greater, smaller or equal to λ^D

e) - Central Banker - "favourite candidate" (1/4)

Your favourite candidate would choose to have an independent Central Bank, choosing the preferences of the central banker (λ^B) in an optimal way. Estimate the inflation and growth rates.

The central banker will set the inflation under discretion. The problem for the central banker is equivalent to question b, why inflation is given by:

$$\pi_t^B = \lambda^B(\bar{x} - \theta_t) + \frac{\lambda^B}{1+\lambda^B}\epsilon_t$$
$$\pi_t^e = \lambda^B(\bar{x} - \theta_t)$$

Output will then be:

$$x_t^B = \theta_t - \frac{1}{1+\lambda^B}\epsilon_t$$

We plug this into the expected loss. This is just like we did in Problem Set 11.

e) - Central Banker - "favourite candidate" (2/4)

We consider the expected loss when keeping the central banker:

$$\mathbb{E}[L(\pi_t^B, x_t^B, \lambda^D)] = \frac{1}{2}\mathbb{E}\Big[(\pi_t^B)^2 + \lambda^D(x_t^B - \bar{x})^2\Big]$$

$$= \frac{1}{2}\mathbb{E}\left[\left(\frac{\lambda^B}{1+\lambda^B}\epsilon_t + \lambda^B(\bar{x} - \theta_t)\right)^2 + \lambda^D\left(\theta_t - \bar{x} - \frac{1}{1+\lambda^B}\epsilon_t\right)^2\right]$$

We use that ϵ and θ are independent random variables with zero mean, why $\mathbb{E}[\epsilon \cdot \theta] = \mathbb{E}[\epsilon] \cdot \mathbb{E}[\theta] = 0$ and $\mathbb{E}[\epsilon \cdot \bar{x}] = \mathbb{E}[\theta \cdot \bar{x}] = 0$

$$= \frac{1}{2}\left[\begin{array}{l} \left(\frac{\lambda^B}{1+\lambda^B}\right)^2 \mathbb{E}[\epsilon_t^2] + (\lambda^B)^2(\bar{x}^2 + \mathbb{E}[\theta_t^2]) \\ +\lambda^D\left[(\bar{x}^2 + \mathbb{E}[\theta_t^2]) + \left(\frac{1}{1+\lambda^B}\right)^2 \mathbb{E}[\epsilon_t^2]\right] \end{array}\right]$$

Since ϵ and θ have zero means we know that $\mathbb{E}[\epsilon^2] = \sigma_\epsilon^2$ and $\mathbb{E}[\theta^2] = \sigma_\theta^2$.

$$= \frac{1}{2}\left[\left(\frac{\lambda^B}{1+\lambda^B}\right)^2 \sigma_\epsilon^2 + (\lambda^B)^2(\bar{x}^2 + \sigma_\theta^2) + \lambda^D\left((\bar{x}^2 + \sigma_\theta^2) + \left(\frac{1}{1+\lambda^B}\right)^2 \sigma_\epsilon^2\right)\right]$$

We now have an expression for the expected loss function

$$\frac{1}{2}\left(\left(\frac{\lambda^B}{1+\lambda^B}\right)^2\sigma_\epsilon^2+(\lambda^B)^2(\bar{x}^2+\sigma_\theta^2)+\lambda^D\left[(\bar{x}^2+\sigma_\theta^2)+\left(\frac{1}{1+\lambda^B}\right)^2\sigma_\epsilon^2\right]\right)$$

We find the derivative wrt. λ^B to find the optimal λ^B

$$\begin{aligned}\frac{\partial\mathbb{E}[L]}{\partial\lambda^B}&=\left(\frac{\lambda^B}{1+\lambda^B}\right)\left(\frac{(1+\lambda^B)-\lambda^B}{(1+\lambda^B)^2}\right)\sigma_\epsilon^2+\lambda^B(\bar{x}^2+\sigma_\theta^2)-\left(\frac{\lambda^D\sigma_\epsilon}{(1+\lambda^B)^3}\right)\\&=\left(\frac{\lambda^B-\lambda^D}{(1+\lambda^B)^3}\right)\sigma_\epsilon+\lambda^B(\bar{x}^2+\sigma_\theta^2)\end{aligned}$$

We define the optimal $\hat{\lambda}^D$ such that $\frac{\partial\mathbb{E}[L]}{\partial\lambda^B}=0$.

We note that $\hat{\lambda}^D\in(0,\lambda^D)$ - Meaning it is greater than zero but below λ^D.

e) - Central Banker - "favourite candidate" (4/4)

Hence, we now know that the ideal central banker has preferences $\lambda^B = \hat{\lambda}^D$ where $0 < \hat{\lambda}^D < \lambda^D$.

We turn back to the output and inflation:

$$\pi_t^B = \lambda^B(\bar{x} - \theta_t) + \frac{\lambda^B}{1 + \lambda^B}\epsilon_t$$

Output will then be:

$$x_t^B = \theta_t - \frac{1}{1 + \lambda^B}\epsilon_t$$

Since, $\lambda^B < \lambda^D$ we note that the inflation bias will be lower for the central banker since the preference for inflation stabilisation is greater.

Meanwhile the output volatility will be higher since $\frac{1}{1+\lambda^B} > \frac{1}{1+\lambda^D}$.

Hence, the government face a trade-off when choosing the optimal central banker since lower λ^B leads to lower pass-through of supply shock to inflation but higher pass-through of supply shocks to output.

f) - Eurozone Peg (c) vs Simple Rule (d) - Set-up

Determine under what conditions the simple rule of c) is preferable to the simple rule of d).

Eurozone peg:

$$\pi_t^{eu} = \frac{1}{1+\lambda^{eu}}\epsilon_t^{eu}$$
$$x_t^{eu} = \theta_t + \frac{1}{1+\lambda^{eu}}\epsilon_t^{eu} - \epsilon_t$$

Simple rule:

$$\pi_t^S = 0$$
$$x_t^S = \theta_t - \epsilon_t$$

Under what conditions does the following hold?

$$\mathbb{E}\left[L(\pi^S, x^S)\right] \geq \mathbb{E}\left[L(\pi^{eu}, x^{eu})\right]$$

f) - Eurozone Peg (c) vs Simple Rule (d) (1/2)

Determine under what conditions the simple rule of c) is preferable to the simple rule of d).

We note that the inflation and output for currency peg (c) is given by:

$$\pi_t^{eu} = \frac{1}{1+\lambda^{eu}}\epsilon_t^{eu}, \qquad x_t^{eu} = \theta_t + \frac{1}{1+\lambda^{eu}}\epsilon_t^{eu} - \epsilon_t$$

The expected loss is

$$\begin{aligned}
\mathbb{E}\left[L^{eu}\right] &= \frac{1}{2}\mathbb{E}\left\{\left(\frac{1}{1+\lambda^{eu}}\epsilon_t^{eu}\right)^2 + \lambda^D\left(\theta_t + \frac{1}{1+\lambda^{eu}}\epsilon_t^{eu} - \epsilon_t - \bar{x}\right)^2\right\} \\
&= \frac{1}{2}\left\{\left(\frac{1}{1+\lambda^{eu}}\right)^2\sigma_{eu}^2 + \lambda^D\left(\sigma_\theta^2 + \left(\frac{1}{1+\lambda^{eu}}\right)^2\sigma_{eu}^2 + \sigma_\epsilon^2 + \bar{x}^2\right)\right\} \\
&= \frac{1}{2}\left\{\frac{1+\lambda^D}{(1+\lambda^{eu})^2}\sigma_{eu}^2 + \lambda^D\left(\sigma_\theta^2 + \sigma_\epsilon^2 + \bar{x}^2\right)\right\}
\end{aligned}$$

f) - Eurozone Peg (c) vs Simple Rule (d) (2/2)

For the simple rule of (d) inflation and output is given by:

$$\pi_t^S = 0, \qquad\qquad x_t^S = \theta_t - \epsilon_t$$

The expected loss is

$$\mathbb{E}\left[L^S\right] = \frac{1}{2}\mathbb{E}\left\{\lambda^D\left(\theta_t + \epsilon_t - \bar{x}\right)^2\right\} = \frac{1}{2}\lambda^D\left(\sigma_\theta^2 + \sigma_\epsilon^2 + \bar{x}^2\right)$$

We compare the two expected losses:

$$\frac{1}{2}\left\{\lambda^D\left(\sigma_\theta^2 + \sigma_\epsilon^2 + \bar{x}^2\right)\right\} \leq \frac{1}{2}\left\{\frac{1+\lambda^D}{(1+\lambda^{eu})^2}\sigma_{eu}^2 + \lambda^D\left(\sigma_\theta^2 + \sigma_\epsilon^2 + \bar{x}^2\right)\right\}$$

$$\mathbb{E}\left[L(\pi^S, x^S)\right] \leq \mathbb{E}\left[L(\pi^{eu}, x^{eu})\right]$$

Hence, the simple rule with $\pi_t = 0$ is strictly dominating for $\sigma_{eu}^2 > 0$.

Note: The strategies yield only identical results if $\sigma_{eu}^2 = 0$ - i.e. only in the case where there are no eurozone shocks.

g) - Discretionary (b) vs Eurozone Peg (c) - Set-up

Determine under what conditions the simple rule of c) is preferable to the simple rule of d).

Discretion:

$$\pi_t^D = \frac{\lambda^D}{1+\lambda^D}\epsilon_t + \lambda^D(\bar{x} - \theta_t)$$
$$x_t^D = \theta_t - \frac{1}{1+\lambda^D}\epsilon_t$$

Eurozone peg:

$$\pi_t^{eu} = \frac{1}{1+\lambda^{eu}}\epsilon_t^{eu}$$
$$x_t^{eu} = \theta_t + \frac{1}{1+\lambda^{eu}}\epsilon_t^{eu} - \epsilon_t$$

Under what conditions does the following hold?

$$\mathbb{E}\left[L(\pi^D, x^D)\right] \leq \mathbb{E}\left[L(\pi^{eu}, x^{eu})\right]$$

g) - Discretionary (b) vs Eurozone Peg (c) (1/2)

Determine under what conditions discretionary policy of b) is preferable to the simple rule of c).

The inflation and output of the discretionary policy is

$$\pi_t^D = \frac{\lambda^D}{1+\lambda^D}\epsilon_t + \lambda^D(\bar{x} - \theta_t), \qquad x_t^D = \theta_t - \frac{1}{1+\lambda^D}\epsilon_t$$

The expected loss, $\mathbb{E}\left[L(\pi^D, x^D)\right]$ is then

$$\begin{aligned}
&\frac{1}{2}\mathbb{E}\left[\left(\frac{\lambda^D}{1+\lambda^D}\epsilon_t + \lambda^D(\bar{x}-\theta_t)\right)^2 + \lambda^D\left(\theta_t - \bar{x} - \frac{1}{1+\lambda^D}\epsilon_t\right)^2\right] \\
&= \frac{1}{2}\left[\left(\frac{\lambda^D}{1+\lambda^D}\right)^2\sigma_\epsilon^2 + (\lambda^D)^2(\bar{x}^2+\sigma_\theta^2) + \lambda^D\left(\sigma_\theta^2 + \bar{x}^2 + \left(\frac{1}{1+\lambda^D}\right)^2\sigma_\epsilon^2\right)\right] \\
&= \frac{1}{2}\left(\frac{(\lambda^D)^2+\lambda^D}{(1+\lambda^D)^2}\sigma_\epsilon^2 + ((\lambda^D)^2+\lambda^D)(\bar{x}_t^2+\sigma_\theta^2)\right) \\
&= \frac{1}{2}\left(\frac{\lambda^D}{1+\lambda^D}\sigma_\epsilon^2 + ((\lambda^D)^2+\lambda^D)(\bar{x}_t^2+\sigma_\theta^2)\right)
\end{aligned}$$

g) - Discretionary (b) vs Eurozone Peg (c) (2/2)

Discretionary policy is preferred to the Eurozone peg when:

$$\mathbb{E}\left[L(\pi^D, x^D)\right] \leq \mathbb{E}\left[L(\pi^{eu}, x^{eu})\right]$$

$$\frac{\lambda^D}{1+\lambda^D}\sigma_\epsilon^2 + ((\lambda^D)^2 + \lambda^D)(\bar{x}_t^2 + \sigma_\theta^2) \leq \frac{1+\lambda^D}{(1+\lambda^{eu})^2}\sigma_{eu}^2 + \lambda^D\left(\sigma_\theta^2 + \sigma_\epsilon^2 + \bar{x}^2\right)$$

$$\frac{\lambda^D}{1+\lambda^D}\sigma_\epsilon^2 + \left(\lambda^D\right)^2(\bar{x}_t^2 + \sigma_\theta^2) \leq \frac{1+\lambda^D}{(1+\lambda^{eu})^2}\sigma_{eu}^2 + \lambda^D\sigma_\epsilon^2$$

$$\left(\lambda^D\right)^2(\bar{x}_t^2 + \sigma_\theta^2) \leq \frac{1+\lambda^D}{(1+\lambda^{eu})^2}\sigma_{eu}^2 + \frac{\left(\lambda^D\right)^2}{1+\lambda^D}\sigma_\epsilon^2$$

- Inflation bias - Additional inflation, which is present under discretion.
- Inflation volatility - Extra volatility induced by Eurozone shocks.
- Output stabilisation - Discretionary policy stabilises output wrt. ϵ, whereas Eurozone peg does not.

Discretionary policy is preferable when the cost of inflation bias is lower than the combined costs of extra inflation and output volatility.

h) - Central Banker Wins

Show that society's welfare is maximized under your favourite candidate's administration.

We evaluate policies b)-d) and compare them with policy e).

We note that

1. Policy b) (discretion) is identical to having a central banker where $\lambda^B = \lambda^D$.
2. Policy d) (Simple rule) is identical to having a central banker where $\lambda^B = 0$.
3. Policy d) weakly dominates policy c).

In question e) we found that $\hat{\lambda}^D \in (0, \lambda^D)$ minimized the loss-function, why policy e) is preferred over b) and d).

Therefore, the central banker is the best solution.

i) - Correlation Between ϵ and ϵ_{eu} (1/2)

Assume now that the supply shocks of Europe and Denmark are correlated (positively or negatively, and denote α the coefficient of correlation). Find the ex ante social loss function for c). For what value of α is it more convenient to adopt the euro?

The coefficient of correlation is denoted α and is defined by:

$$\alpha = \frac{cov(\epsilon_t, \epsilon_t^{eu})}{\sigma_\epsilon \sigma_{eu}} = \frac{\mathbb{E}\left[\left(\epsilon_t - \mathbb{E}[\epsilon_t]\right)\left(\epsilon_t^{eu} - \mathbb{E}[\epsilon_t^{eu}]\right)\right]}{\sigma_\epsilon \sigma_{eu}} = \frac{\mathbb{E}[\epsilon \epsilon_t^{eu}]}{\sigma_\epsilon \sigma_{eu}}$$

We can then rewrite it in the following manner:

$$\mathbb{E}[\epsilon \epsilon_t^{eu}] = \alpha \sigma_\epsilon \sigma_{eu}$$

i) - Correlation Between ϵ and ϵ_{eu} (2/2)

$$\mathbb{E}\left[L^{eu}\right] = \frac{1}{2}\mathbb{E}\left[\left(\frac{1}{1+\lambda^{eu}}\epsilon_t^{eu}\right)^2 + \lambda^D\left(\theta_t + \frac{1}{1+\lambda^{eu}}\epsilon_t^{eu} - \epsilon_t - \bar{x}\right)^2\right]$$

$$= \frac{1}{2}\Bigg[\underbrace{\frac{1+\lambda^D}{(1+\lambda^{eu})^2}\sigma_{eu}^2 + \lambda^D\left(\sigma_\theta^2 + \sigma_\epsilon^2 + \bar{x}^2\right)}_{\text{Same as before}} - \frac{\lambda^D \cdot 2}{1+\lambda^{eu}}\mathbb{E}[\epsilon\epsilon_t^{eu}]\Bigg]$$

$$= \frac{1}{2}\left[\frac{1+\lambda^D}{(1+\lambda^{eu})^2}\sigma_{eu}^2 + \lambda^D\left(\sigma_\theta^2 + \sigma_\epsilon^2 + \bar{x}^2\right) - \frac{\lambda^D \cdot 2}{1+\lambda^{eu}}\alpha\sigma_\epsilon\sigma_{eu}\right] \quad (5)$$

We see from (5) that $\alpha = 1$ (perfect correlation) minimizes the loss.[1]

- When the shocks are perfectly correlated, the monetary policy of the Eurozone, which Denmark adopts, will help stabilising the Danish economy wrt. in response to domestic shocks.
- If the shocks are negatively correlated, the peg will exacerbate the volatility. The monetary policy in Denmark will be expansionary when it should be contractionary.

[1] The correlation coefficient is by definition restricted to be $\alpha \in [-1, 1]$.